### 1. Introduction

IN a recent paper,[1] the writer calculated the transverse modes of vibration of a spinning, elastic membrane under the assumption that the membrane was held between two hubs in such a manner that radial but not transverse displacements could occur at the hubs (partial clamping). The static stress field was taken to be that of a freely spinning solid disk, which permitted the equation of motion to be reduced to a hypergeometric equation. In Ref. 1, it was remarked that, if the membrane were fully clamped at the hubs, i.e., if neither radial nor transverse displacements could occur there, the static stress field would be more complicated, and the reduced equation of motion would be Heun's equation—an ordinary second-order differential equation with four regular singular points.

For the axisymmetric modes of a spinning, fully clamped membrane, one of the singularities in Heun's equation disappears, and Legendre's equation is obtained. Below, a simple change of scale is given which reduces this equation and its boundary conditions to a corresponding problem for a partially clamped membrane.

### 2. Equation of Motion

In the notation of Ref. 1, the differential equation governing axisymmetric vibrations of infinitesimal amplitude is

$$\frac{\partial}{\partial r}\left(r\sigma_r \frac{\partial w}{\partial r}\right) - \rho r \frac{\partial^2 w}{\partial t^2} = 0 \tag{1}$$

For a membrane of radius $b$ fully clamped to a hub of radius $a$, the radial stress is [see Eq. (53) of Ref. 3, for example]

$$\sigma_r = \frac{3+\nu}{8}\rho\Omega^2(b^2 - r^2)\left(1 + \epsilon\frac{a^2}{r^2}\right) \tag{2}$$

where

$$\epsilon = \frac{1-\nu}{3+\nu}\left[\frac{3+\nu-\lambda^2(1+\nu)}{1+\nu+\lambda^2(1-\nu)}\right] \tag{3}$$

where $\nu$ is Poisson's ratio, and $\lambda = a/b$. (For a partially clamped membrane, the radial stress is obtained by setting $\epsilon = 0$.) The boundary conditions associated with Eq. (1) are

$$r = a:\ w = 0 \qquad r = b:\ w = \text{finite} \tag{4}$$

Substituting Eq. (2) into Eq. (1) and setting

$$\frac{w}{b} = y(x)\sin\omega t \qquad x = \frac{1-(r/b)^2}{1+\lambda^2\epsilon} \tag{5}$$

Legendre's differential equation (with singularities at $x = 0$ and $x = 1$) is obtained:

$$\frac{d}{dx}\left[x(1-x)\frac{dy}{dx}\right] + \frac{2\omega^2}{(3+\nu)\Omega^2}y = 0 \tag{6}$$

together with the boundary conditions

$$x = 0:\ y = \text{finite} \qquad x = \frac{1-\lambda^2}{1+\lambda^2\epsilon}:\ y = 0 \tag{7}$$

A comparison of Eqs. (5–7) with those of either Johnson[2] or the writer[1] for the axisymmetric modes of a spinning, partially clamped membrane shows that the two problems are equivalent provided that $\Lambda$, the ratio of hub to membrane radii for the partially clamped membrane, be related to $\lambda$ of the fully clamped membrane by the expression

$$\Lambda^2 = \lambda^2\left(\frac{1+\epsilon}{1+\lambda^2\epsilon}\right) \tag{8}$$

Thus the frequency curves presented by the author[1] and the discussion by Johnson[2] of the case $\Lambda^2 \ll 1$ are directly applicable to the case of a fully clamped membrane.

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# Oblique Detonation Waves

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**The two-dimensional, steady-flow equations for oblique detonation waves are developed, and solutions for the jump conditions are presented. A proof is presented that the corresponding Chapman-Jouguet condition for oblique detonation waves occurs when the downstream velocity component, normal to the wave, is sonic.**

### Nomenclature

$a$ = local speed of sound
$c_p$ = specific heat at constant pressure
$M$ = Mach number defined as $V/a$
$p$ = pressure
$Q$ = heat addition per unit mass of fluid
$R$ = gas constant
$T$ = absolute temperature
$V$ = velocity
$\gamma$ = ratio of specific heats
$\delta$ = flow deflection angle
$\rho$ = density
$\sigma$ = detonation wave angle

***Subscripts***

$n$ = component normal to the wave surface
1 = state of the reactant gas upstream of the wave
2 = state of the product gas downstream of the wave

### Introduction

THE development of high-temperature steady-flow combustion tunnels and the quest for finding methods to permit breathing propulsion systems to obtain very high speeds has generated interest in oblique detonation waves. A detonation wave is an exothermal wave that moves supersonically with respect to the reactant gas. The thickness of the wave and whether it can be treated as a shock wave followed by chemical reactions is a question of wave structure. Only those waves whose thickness is small compared to the extent of the wave surface are considered here, so that the wave may be treated as a discontinuity in the flow.

Experiments on oblique detonation-like waves have been reported by Gross and Chinitz[1] and Rhodes et al.[2] Samaras[3]

Received January 28, 1963. This study was supported by the Air Force Office of Scientific Research under Grants AF AFOSR-48-63 and AF 49(638)-15. It is a pleasure to acknowledge the assistance of W. Chinitz, L. Bohrer, and K. Foreman for help in this work, which was done originally in 1959 under support by the Air Force Office of Scientific Research.

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previously has discussed oblique detonation waves and some of their jump properties. Siestrunck et al.[4] and Rutkowski and Nicholls[5] have suggested a polar form of displaying properties of oblique detonations. Unfortunately, these forms of solution are cumbersome and awkward to use. It has been found to be considerably simpler and more useful to employ the initial stream Mach number, the amount of heat addition, and the wave angle as the independent variables.

### Equations

Consider an oblique detonation wave where the flow vectors behave as shown in Fig. 1. Conservation of mass across the wave is expressed by

$$\rho_1 V_1 \sin\sigma = \rho_2 V_2 \sin(\sigma - \delta) \tag{1}$$

Conservation of momentum normal and tangential to the wave is expressed by

$$p_1 = \rho_1 V_1^2 \sin^2\sigma = p_2 + \rho_2 V_2^2 \sin^2(\sigma - \delta) \tag{2}$$

$$\rho_1 V_1^2 \sin\sigma \cos\sigma = \rho_2 V_2^2 \sin(\sigma - \delta)\cos(\sigma - \delta) \tag{3}$$

The energy equation is simply

$$c_{p1} T_1 + (V_1^2/2) + Q = C_{p2} T_2 + (V_2^2/2) \tag{4}$$

and the equation of state is adopted:

$$p = \rho R T \tag{5}$$

Equations (1–5) represent five equations for the five unknowns, $\rho_2$, $V_2$, $p_2$, $T_2$, and $\delta$ in terms of the initial flow conditions, the wave angle $\sigma$, and the quantity of heat addition $Q$. With the assumption that $C_{p_1} = C_{p_2}$, these equations can be solved conveniently for the density ratio across the wave.

Thus,

$$\frac{\rho_1}{\rho_2} = \frac{1 + \gamma M_1^2 \sin^2\sigma \pm \{[1 + \gamma M_1^2 \sin^2\sigma]^2 - (\gamma + 1) M_1^2 \sin^2\sigma\,[2 + (2Q/c_p T_1) + (\gamma - 1) M_1^2 \sin^2\sigma]\}^{1/2}}{(\gamma + 1) M_1^2 \sin^2\sigma} \tag{6}$$

When $Q$ is zero, it readily can be shown that this reduces simply to the density jump across an oblique shock wave. The form of Eq. (6) is quadratic, implying that under some initial conditions there are two real final states. This is directly analogous to normal detonation wave theory, where analytically there exist both strong and weak detonation solutions such as discussed in Ref. 1. When the square root in Eq. (6) becomes zero, the solution is singular, corresponding to the well-known Chapman-Jouguet situation. Thus, oblique detonation waves should exhibit the Chapman-Jouguet condition.

The remaining flow variables readily can be solved in terms of the initial flow conditions and the density ratio given by Eq. (6). Thus,

$$V_2/V_1 = \{1 - \sin^2\sigma[1 - (\rho_1/\rho_2)^2]\}^{1/2} \tag{7}$$

$$p_2/p_1 = 1 + \gamma M_1^2 \sin^2\sigma \cdot [1 - (\rho_1/\rho_2)] \tag{8}$$

$$T_2/T_1 = (p_2/p_1)(\rho_1/\rho_2) \tag{9}$$

$$\delta = \sigma - \cos^{-1}[(V_1/V_2)\cos\sigma] \tag{10}$$

and, from the definition of Mach number,

$$M_2/M_1 = (V_2/V_1)(T_1/T_2)^{1/2} \tag{11}$$

It now will be proved that the Chapman-Jouguet situation for oblique waves occurs when the downstream velocity component normal to the wave is sonic, namely, $M_{2n} = 1$. The Chapman-Jouguet condition occurs when the strong and weak solutions merge to the singular solution, which means that the radical in Eq. (6) is zero. For this condition, Eq. (6) reduces to

$$\frac{\rho_1}{\rho_2} = \frac{1 + \gamma M_1^2 \sin^2\sigma}{(\gamma + 1) M_1^2 \sin^2\sigma} \tag{12}$$

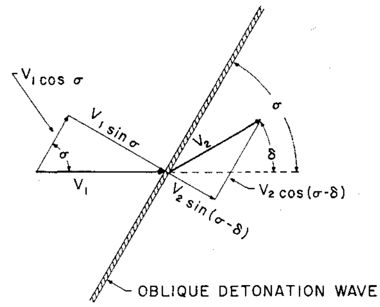



Fig. 1

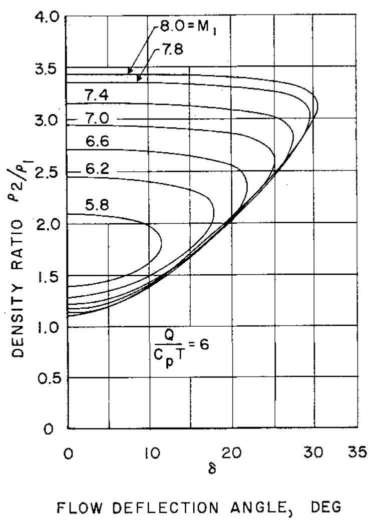



Fig. 2

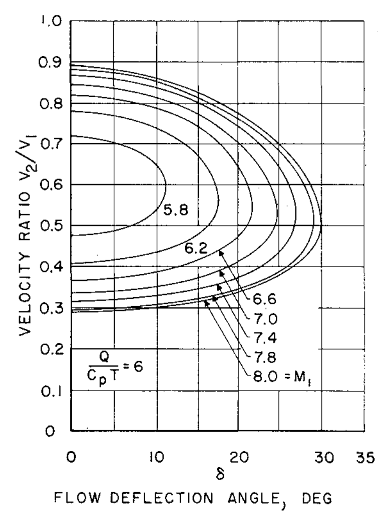



Fig. 3

An identity, using the definition of Mach number, is

$$\frac{M_2^2 \sin^2(\sigma - \delta)}{M_1^2 \sin^2\sigma} = \left(\frac{V_2}{V_1}\right)^2 \left(\frac{T_1}{T_2}\right) \frac{\sin^2(\sigma - \delta)}{\sin^2\sigma} \tag{13}$$

Using Eqs. (13, 1, and 5), one obtains

$$\frac{M_2^2 \sin^2(\sigma - \delta)}{M_1^2 \sin^2\sigma} = \left(\frac{\rho_1}{\rho_2}\right)\left(\frac{p_1}{p_2}\right) \tag{14}$$

Using Eq. (8) with (14),

$$\frac{M_2^2 \sin^2(\sigma - \delta)}{M_1^2 \sin^2\sigma} = \frac{\rho_1/\rho_2}{1 + \gamma M_1^2 \sin^2\sigma\,[1 - (\rho_1/\rho_2)]} \tag{15}$$

Equation (15) together with (12) then gives the desired result, namely,

Fig. 4

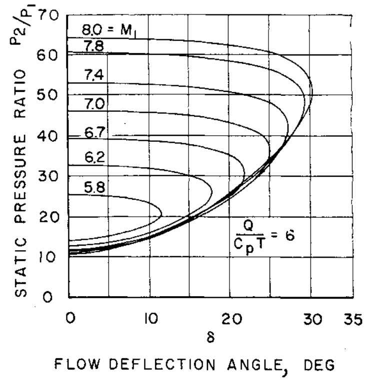



Fig. 5

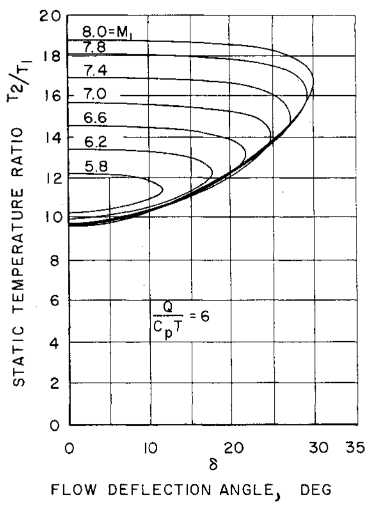



Fig. 6

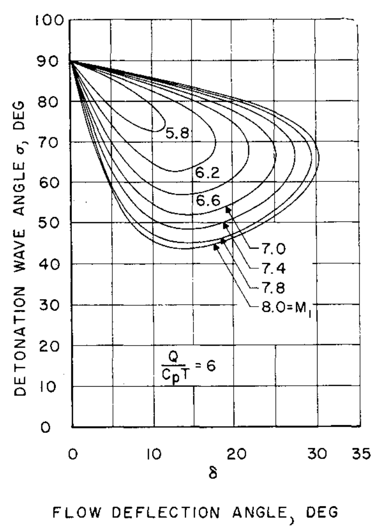
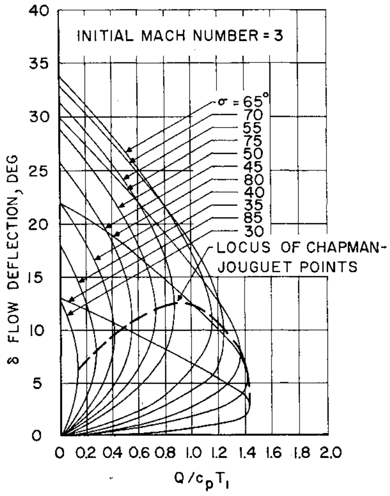



Fig. 7




$$M_2^2 \sin^2(\sigma - \delta) = 1 \tag{16}$$

or

$$M_{2n} = 1 \tag{16a}$$

### Results

Some typical plots of the solutions of these oblique detonation wave equations are shown in Figs. 2–7. As with normal detonation waves, for a given value of $Q/c_pT_1$, there is a minimum value of $M_1$, the Chapman-Jouguet condition, which permits real values for the radical in Eq. (6). Below this minimum, the assumption of steady flow is violated. Figures 2–6 represent Eqs. (6–10). Figure 7 represents the wave behavior at a fixed Mach number, $M_1 = 3$, as the heat addition parameter (sometimes called Damkohler second parameter) is varied. The locus of the Chapman-Jouguet points is shown. The region above the *C-J* line in Fig. 7 represents the strong detonation solutions.

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# Forced-Convection Heat Transfers with Time-Dependent Surface Temperatures

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AEROTHERMODYNAMICISTS frequently calculate the rate of forced-convection heat transfer to a surface with time-dependent temperature, e.g., to the external surface of a high-speed vehicle subjected to thermal radiation from a nuclear explosion. The applicability of the equations developed for steady-state forced-convection heat transfer has been studied independently and simultaneously by Knuth and Bussell[1] and by Sparrow and Gregg;[2] results of a related study have been published recently by Goodman.[3] In the first paper, the geometrical aspects of the problem are simplified by considering Couette flow; initial conditions are specified, and heat transfer rates for subsequent values of time are computed. In the second paper, the temporal aspects are simplified by neglecting the past history of the system and examining the departure of the instantaneous conditions from quasi-steady conditions; computations are made for laminar boundary layer flow of a compressible fluid with Prandtl number of 0.72. In the third paper, the geometrical aspects are simplified by using an integral method; equations are developed for laminar boundary layer flow of an incompressible fluid with arbitrary Prandtl number. The purpose of the present note is to show how these three studies com-

Received January 25, 1963.

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